Lecture 7 Highlights
Phys 402

The Hyperfine Interaction in Hydrogen

We considered the hyperfine interaction between the magnetic moment of the proton and that of the electron in the hydrogen atom.

The proton has a magnetic moment due to the intrinsic spin, orbital motion of its quark constituents, and the quark-gluon plasma. It is given by:

$$\vec{\mu}_p = \frac{ge}{2m_p}\vec{S}_p,$$

where $g \cong 5.59$, e is the electronic charge, m_p is the mass of the proton, and $\vec{S}_p$ is its spin angular momentum. The fact that g has such an odd value for the proton (as opposed to 2.00 for the electron) implies that it has internal structure. Note that the proton is a spin-1/2 particle, just like an electron ($\vec{\mu}_e = -\frac{e}{m_e}\vec{S}_e$). Its spin lives on a 2-state ladder, with steps separated by $\hbar$, just like the electron. In the hydrogen atom the magnetic field generated by this magnetic moment interacts with the magnetic moment of the electron to give rise to the hyperfine perturbing Hamiltonian:

$$\mathrm{H}_{HF} = -\vec{\mu}_e \bullet \vec{B}_{dip},$$

where the magnetic field due to the proton's dipole moment is given by Griffiths E+M book, Eq. 5.90:

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{\mu}_p \bullet \hat{r})\hat{r} - \vec{\mu}_p\right] + \frac{2\mu_0}{3}\vec{\mu}_p\delta^3(\vec{r}),$$

where the dipole lies at the origin of the coordinate system. The last term comes from the infinitesimal dipole at the proton location. (The calculation assumes a uniformly magnetized sphere rotating at the origin. One takes the limit as the radius of the sphere goes to zero. The internal field diverges in such a way that the magnetic moment of the infinitesimal sphere remains finite at $\vec{\mu}_p$. This process leads to the delta-function term. See David J. Griffiths, "Hyperfine splitting in the ground state of hydrogen," *Am. J. Phys.* 50, 698 (1982), posted on the class web site)

Note that we could also take the perspective of the proton and evaluate the magnetic field created by the electron magnetic moment at the proton location. Hence we expect that the final expression for the perturbing Hamiltonian to be symmetric with respect to the proton and electron.

In order to keep things simple we will only consider Hydrogen atoms in the $\vec{L} = 0$ state. This eliminates the possibility of spin-orbit interaction with the nuclear spin. It is also the most interesting case anyway.

Evaluating the first order correction to the energy of the hydrogen atom yields:

$$E_n^1 = \left\langle \psi_n^0 \middle| \mathrm{H}_{HF} \middle| \psi_n^0 \right\rangle$$

$$= \frac{\mu_0 g e^2}{8\pi m_e m_p}\left\langle \psi_n^0 \middle| \frac{3(\vec{S}_p \bullet \hat{r})(\vec{S}_e \bullet \hat{r}) - \vec{S}_e \bullet \vec{S}_p}{r^3} \middle| \psi_n^0 \right\rangle + \frac{\mu_0 g e^2}{3m_e m_p}\left\langle \psi_n^0 \middle| \vec{S}_e \bullet \vec{S}_p \delta^3(\vec{r}) \middle| \psi_n^0 \right\rangle$$

Remember that "n" in the wavefunction subscript represents a list of quantum numbers, in general. Here we are using the standard un-perturbed Hydrogen atom wavefunctions $|\psi_n^0\rangle \sim |\ell\ m_\ell\rangle |s\ m_s\rangle$.

If we specialize the case of zero orbital angular momentum, $\ell = 0$, for the un-perturbed states, the first term above is zero (see problem 7.31). The second term simplifies because of the delta function, and we have:

$$E_{n,0,0}^1 = \frac{\mu_0 g e^2}{3 m_e m_p} \left\langle \vec{S}_e \bullet \vec{S}_p \right\rangle \left| \psi_{n,0,0}^0(\vec{r} = 0) \right|^2,$$

where now n represents the principal quantum number in the hydrogen atom. One finds from Eq. (4.89) that $\left| \psi_{n,0,0}^0(0) \right|^2 = \frac{1}{n^3} \frac{1}{\pi a^3}$, where a is the Bohr radius. To evaluate $\left\langle \vec{S}_e \bullet \vec{S}_p \right\rangle$ we define a total angular momentum vector $\vec{S} = \vec{S}_e + \vec{S}_p$, just as we did before to evaluate the spin-orbit perturbation. Note that $\vec{L} = 0$ here by assumption. Hence $\vec{S}$ is the total angular moment of the Hydrogen atom and is therefore a constant of the motion in the absence of an external torque (i.e. an external magnetic field). Through the same manipulations used to evaluate $\vec{L} \cdot \vec{S}$ in the spin-orbit perturbation, we arrive at the following result;

$$\left\langle \vec{S}_e \bullet \vec{S}_p \right\rangle = \frac{1}{2} \left(\left\langle S^2 \right\rangle - \left\langle S_e^2 \right\rangle - \left\langle S_p^2 \right\rangle \right).$$

Note that the electron and proton are both spin-1/2 particles and so[1] $\left\langle S_e^2 \right\rangle = \left\langle S_p^2 \right\rangle = s(s+1)\hbar^2 = 3\hbar^2/4$, and

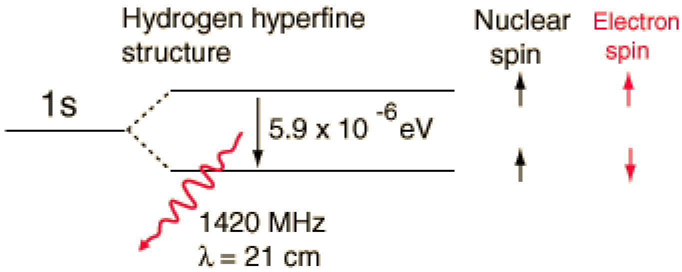
$$\left\langle \vec{S}_e \bullet \vec{S}_p \right\rangle = \frac{1}{2} \left(\left\langle S^2 \right\rangle - \frac{3\hbar^2}{2} \right)$$

We treated the total spin of two spin-1/2 particles in the Discussion 3 and found that two ladders of states are possible, that of $s = 1$ (the 3-state Triplet with $\left\langle S^2 \right\rangle = 2\hbar^2$) and $s = 0$ (the 1-state Singlet with $\left\langle S^2 \right\rangle = 0\hbar^2$). This yields two possible values for the first-order corrected energy:

$$E_{n,0,0}^1 = \frac{\mu_0 g e^2}{3 m_e m_p} \frac{\hbar^2}{\pi n^3 a^3} \begin{cases} 1/4 & \text{TRIPLET} \\ -3/4 & \text{SINGLET} \end{cases}$$

Consider the case of $n = 1$, which is the ground state of Hydrogen (1s). This state is now split into two hyperfine-split states as shown in the diagram.

[1] What wavefunction do we use to calculate $\langle S_p^2 \rangle$? We can augment the hydrogen atom wavefunction with a proton spinor ket as $\psi_n^0 = |n\ \ell\ m_\ell\rangle |s_e\ m_{se}\rangle |s_p\ m_{sp}\rangle$, which is a direct product state. The proton spinor ket $|s_p\ m_{sp}\rangle$ introduces the 2D Hilbert space of the proton spin, and expands the Hilbert space of the hydrogen atom by a factor of 2.



The energy splitting is only about 6 μeV, compared to the ground state binding energy of 13.6 eV. The upper state has a lifetime of about 10^{15} seconds, or about 10^8 years. When the atom makes a transition from the triplet state to the singlet state, it gives off radiation of frequency 1.420 GHz, with a wavelength of about 21 cm. This radiation can propagate through clouds of dust in the galaxy. From measurements of the Doppler shift of this radiation, the spiral structure of our galaxy was deduced. This transition photon was also used as the standard of length and time in the "post card" attached to the Pioneer 10 spacecraft and the phonograph record attached to the Voyager spacecraft.

Note that the picture of the orientation of the Nuclear spin and Electron spin in the above figure is somewhat deceiving. The actual states are described by the triplet and singlet spin wavefunctions, and can not be understood in terms of the incomplete "un-coupled" representation illustrated with the black and red arrows.